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Classification of local realistic theories

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Abstract
Recently, it has been shown that an explicit local realistic model for the values of a correlation function, given in a two-setting Bell experiment (two-setting model), works only for the specific set of settings in the given experiment, but cannot construct a local realistic model for the values of a correlation function, given in a *continuous-infinite* settings Bell experiment (infinite-setting model), even though there exist two-setting models for all directions in space. Hence, the two-setting model does not have the property which the infinite-setting model has. Here, we show that an explicit two-setting model cannot construct a local realistic model for the values of a correlation function, given in an *only discrete-three* settings Bell experiment (three-setting model), even though there exist two-setting models for the three measurement directions chosen in the given three-setting experiment. Hence, the two-setting model does not have the property which the three-setting model has.

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1. Introduction

There is much research about local realism [1–4]. The locality condition says that a result of measurement pertaining to one system is independent of any measurement performed simultaneously at a distance on another system. Quantum mechanics does not allow a local realistic interpretation. Certain quantum predictions violate Bell inequalities [2], which are conditions that a local realistic theory must satisfy. Experimental efforts (Bell experiment) of a violation of local realism can be seen in [5–7]. Other types of inequalities are given in [8, 9]. Bell inequalities with settings other than spin polarizations can be seen in [10].

Here, we consider the Bell experiment for a system described by multipartite states in the case where n-dichotomic observables are measured per site. If n is two, we consider a two-setting Bell experiment. If n is three, we consider a three-setting Bell experiment and so on.

Recently, it has been shown [11] that an explicit local realistic model for the values of a correlation function, given in a two-setting Bell experiment (two-setting model), works only for the specific set of settings in the given experiment, but cannot construct a local realistic model for the values of a correlation function, given in a Bell experiment with continuous-infinite settings lying in a plane (plane-infinite-setting model), even though there exist two-setting models for all directions in the plane. Therefore, the property of the two-setting model is different from the property of the plane-infinite-setting model.

Further, in a specific type of quantum states, it was shown [12] that the Bell inequality with the assumption of the existence of a local realistic model which is rotationally invariant (sphere-infinite-setting model) disproves the two-setting model stronger than Bell inequality with the assumption of the existence of a local realistic model which is rotationally invariant with respect to a plane (i.e. plane-infinite-setting model). Therefore, the property of the two-setting model is different from the property of the sphere-infinite-setting model. Also we see that the property of the plane-infinite-setting model is different from the property of the sphere-infinite-setting model.

We thus see that there is a division among the measurement settings, those that admit local realistic models which are rotationally invariant (sphere-infinite-setting model), those that admit local realistic models which are rotationally invariant with respect to a plane (plane-infinite-setting model), and those that do not (e.g. two-setting model). This is another manifestation of the underlying contextual nature of local realistic theories of quantum experiments.

In this paper, we shall show that the two-setting model cannot construct a local realistic model for the values of a correlation function, given in a three-setting Bell experiment (three-setting model), even though there exist two-setting models for the three measurement directions chosen in the given three-setting experiment. The property of the two-setting model is different from the property of the three-setting model. To this end we derive a generalized Bell inequality for N qubits, which involves three-setting for each of the local measuring apparatuses. The inequality forms a necessary condition for the existence of the three-setting model. Although the inequality involves three settings, it can be experimentally tested using two orthogonal local measurement settings. This is a direct consequence of the assumed form of the rotationally invariant correlation like (2). We see our generalized Bell inequality with the assumption of the existence of the three-setting model disproves the two-setting model for the actually measured values of the correlation function.

Our result provides classification of local realistic theories. In order to say that some model is different from another model, we need a criterion. Our criterion is as follows. *We may say that model (A1) is different from model (A2) if model (A1) does not have the property which model (A2) has*. We shall stand to this approach.

Then, we can see four types of models at least. First, there is two-setting model. It is explicitly constructed by standard two-setting Bell inequalities [13]. However, this model is disproved by several generalized Bell inequalities. The patterns of the disqualification are different from each other. Therefore, one furthermore has three different types of models. These are the three-setting model, plane-infinite-model and sphere-infinite-model, as we mentioned above.

2. Multipartite three-setting generalized Bell inequality

Assume that we have a set of N spins $\frac{1}{2}$. Each of them is in a separate laboratory. As is well known, the measurements (observables) for such spins are parameterized by a unit vector $\vec{n}_j$, $j = 1, 2, \ldots, N$ (direction along which the spin component is measured). The results of measurements are ± 1 (in $\hbar/2$ unit). One can introduce the 'Bell' correlation function, which is the average of the product of the local results:

$$E(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \langle r_1(\vec{n}_1) r_2(\vec{n}_2) \ldots r_N(\vec{n}_N)\rangle_{\mathrm{avg}}, \tag{1}$$

where $r_j(\vec{n}_j)$ is the local result, ± 1, which is obtained if the measurement direction is set at $\vec{n}_j$. If the experimental correlation function admits a rotationally invariant tensor structure familiar from quantum mechanics, we can introduce the following form:

$$E(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \hat{T} \cdot (\vec{n}_1 \otimes \vec{n}_2 \otimes \cdots \otimes \vec{n}_N), \tag{2}$$

where $\otimes$ denotes the tensor product, $\cdot$ the scalar product in R^{3N} and $\hat{T}$ is the correlation tensor given by

$$T_{i_1 \ldots i_N} \equiv E\left(\vec{x}_1^{(i_1)}, \vec{x}_2^{(i_2)}, \ldots, \vec{x}_N^{(i_N)}\right), \tag{3}$$

where $\vec{x}_j^{(i_j)}$ is a unit directional vector of the local coordinate system of the jth observer; $i_j = 1, 2, 3$ gives the full set of orthogonal vectors defining the local Cartesian coordinates. That is, the components of the correlation tensor are experimentally accessible by measuring the correlation function at the directions given by the basis vectors of local coordinate systems. Obviously the assumed form of (2) implies rotational invariance, because the correlation function is a scalar. Rotational invariance simply states that the value of $E(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N)$ cannot depend on the local coordinate systems used by the N observers. Assume that one knows the values of all 3^N components of the correlation tensor, $T_{i_1 \ldots i_N}$, which are obtainable by performing specific 3^N measurements of the correlation function, compare equation (3). Then, with the use of formula (2) one can reproduce the value of the correlation functions for all other possible sets of local settings.

Using this rotationally invariant structure of the correlation function, we shall derive a necessary condition for the existence of a local realistic model for the values of the experimental correlation function (2) given in a three-setting Bell experiment.

If the correlation function is described by a local realistic theory, then the correlation function must be simulated by the following structure:

$$E_{\mathrm{LR}}(\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_N) = \int \mathrm{d}\lambda \rho(\lambda) I^{(1)}(\vec{n}_1, \lambda) I^{(2)}(\vec{n}_2, \lambda) \cdots I^{(N)}(\vec{n}_N, \lambda), \tag{4}$$

where λ is some local hidden variable, $\rho(\lambda)$ is a probabilistic distribution, and $I^{(j)}(\vec{n}_j, \lambda)$ is the predetermined 'hidden' result of the measurement of the dichotomic observable $\vec{n} \cdot \vec{\sigma}$ with values ± 1.

Let us parametrize the three unit vectors in the plane defined by $\vec{x}_j^{(1)}$ and $\vec{x}_j^{(2)}$ in the following way

$$\vec{n}_j\left(\alpha_j^{l_j}\right) = \cos \alpha_j^{l_j} \vec{x}_j^{(1)} + \sin \alpha_j^{l_j} \vec{x}_j^{(2)}, \qquad j = 1, 2, \ldots, N. \tag{5}$$

The phases $\alpha_j^{l_j}$ that experimentalists are allowed to set are chosen as

$$\alpha_j^{l_j} = (l_j - 1)\pi/3, \qquad l_j = 1, 2, 3. \tag{6}$$

We shall show that the scalar product of the 'three-setting' the local realistic correlation function

$$E_{\mathrm{LR}}\left(\alpha_1^{l_1}, \alpha_2^{l_2}, \ldots, \alpha_N^{l_N}\right) = \int \mathrm{d}\lambda \rho(\lambda) I^{(1)}\left(\alpha_1^{l_1}, \lambda\right) I^{(2)}\left(\alpha_2^{l_2}, \lambda\right) \cdots I^{(N)}\left(\alpha_N^{l_N}, \lambda\right), \tag{7}$$

with the rotationally invariant correlation function, that is

$$E\left(\alpha_1^{l_1}, \alpha_2^{l_2}, \ldots, \alpha_N^{l_N}\right) = \hat{T} \cdot \vec{n}_1\left(\alpha_1^{l_1}\right) \otimes \vec{n}_2\left(\alpha_2^{l_2}\right) \otimes \cdots \otimes \vec{n}_N\left(\alpha_N^{l_N}\right), \tag{8}$$

is bounded by a specific number dependent on $\hat{T}$. We define the scalar product (E_{LR}, E) as follows. We see that the maximal possible value of (E_{LR}, E) is bounded as:

$$\begin{aligned}(E_{\mathrm{LR}}, E) = &\sum_{l_1=1,2,3}\sum_{l_2=1,2,3}\cdots\sum_{l_N=1,2,3} E_{\mathrm{LR}}\big(\alpha_1^{l_1}, \alpha_2^{l_2}, \ldots, \alpha_N^{l_N}\big)\\ &E\big(\alpha_1^{l_1}, \alpha_2^{l_2}, \ldots, \alpha_N^{l_N}\big) \leqslant 2^N T_{\max},\end{aligned} \tag{9}$$

where $T_{\max}$ is the maximal possible value of the correlation tensor component, i.e.

$$T_{\max} \equiv \max_{\beta_1,\beta_2,\ldots,\beta_N} E(\beta_1, \beta_2, \ldots, \beta_N), \tag{10}$$

where β_j is some angle.

A necessary condition for the existence of the 'three-setting' local realistic description E_{LR} of the experimental correlation function

$$E\big(\alpha_1^{l_1}, \alpha_2^{l_2}, \ldots, \alpha_N^{l_N}\big) = E\big(\vec{n}_1\big(\alpha_1^{l_1}\big), \ldots, \vec{n}_N\big(\alpha_N^{l_N}\big)\big), \tag{11}$$

that is for E_{LR} to be equal to E for the three measurement directions, is that one has $(E_{\mathrm{LR}}, E) = (E, E)$. This implies the possibility of modeling E by the 'three-setting' local realistic correlation function E_{LR} given in (7) with respect to the three measurement directions. If we have $(E_{\mathrm{LR}}, E) < (E, E)$, then the experimental correlation function cannot be explainable by the three-setting local realistic model. (Note that, due to the summation in (9), we are looking for the three-setting model.)

In what follows, we derive the upper bound (9). Since the local realistic model is an average over λ, it is enough to find the bound of the following expression:

$$\sum_{l_1=1,2,3}\cdots\sum_{l_N=1,2,3} I^{(1)}\big(\alpha_1^{l_1}, \lambda\big)\cdots I^{(N)}\big(\alpha_N^{l_N}, \lambda\big)\sum_{i_1,i_2,\ldots,i_N=1,2} T_{i_1 i_2\ldots i_N} c_1^{i_1} c_2^{i_2}\ldots c_N^{i_N}, \tag{12}$$

where

$$\big(c_j^1, c_j^2\big) = \big(\cos\alpha_j^{l_j}, \sin\alpha_j^{l_j}\big), \tag{13}$$

and

$$T_{i_1 i_2\ldots i_N} = \hat{T}\cdot\big(\vec{x}_1^{(i_1)}\otimes\vec{x}_2^{(i_2)}\otimes\cdots\otimes\vec{x}_N^{(i_N)}\big), \tag{14}$$

compare (2) and (3).

Let us analyze the structure of this sum (12). Note that (12) is a sum, with coefficients given by $T_{i_1 i_2\ldots i_N}$, which is a product of the following sums:

$$\sum_{l_j=1,2,3} I^{(j)}\big(\alpha_j^{l_j}, \lambda\big)\cos\alpha_j^{l_j}, \tag{15}$$

and

$$\sum_{l_j=1,2,3} I^{(j)}\big(\alpha_j^{l_j}, \lambda\big)\sin\alpha_j^{l_j}. \tag{16}$$

We deal here with sums, or rather scalar products of $I^{(j)}\big(\alpha_j^{l_j}, \lambda\big)$ with two orthogonal vectors. One has

$$\sum_{l_j=1,2,3}\cos\alpha_j^{l_j}\sin\alpha_j^{l_j} = 0, \tag{17}$$

because,

$$2\times\sum_{l_j=1,2,3}\cos\alpha_j^{l_j}\sin\alpha_j^{l_j} = \sum_{l_j=1,2,3}\sin 2\alpha_j^{l_j} = \mathrm{Im}\left(\sum_{l_j=1,2,3}\mathrm{e}^{\mathrm{i}2\alpha_j^{l_j}}\right). \tag{18}$$

Since $\sum_{l_j=1}^{3}\mathrm{e}^{\mathrm{i}(l_j-1)(2/3)\pi} = 0$, the last term vanishes.

Please note

$$\sum_{l_j=1}^{3}\left(\cos\alpha_j^{l_j}\right)^2=\sum_{l_j=1}^{3}\frac{1+\cos 2\alpha_j^{l_j}}{2}=3/2 \tag{19}$$

and

$$\sum_{l_j=1}^{3}\left(\sin\alpha_j^{l_j}\right)^2=\sum_{l_j=1}^{3}\frac{1-\cos 2\alpha_j^{l_j}}{2}=3/2, \tag{20}$$

because,

$$\sum_{l_j=1,2,3}\cos 2\alpha_j^{l_j}=\mathrm{Re}\left(\sum_{l_j=1,2,3}\mathrm{e}^{\mathrm{i}2\alpha_j^{l_j}}\right). \tag{21}$$

Since $\sum_{l_j=1}^{3}\mathrm{e}^{\mathrm{i}(l_j-1)(2/3)\pi}=0$, the last term vanishes.

The normalized vectors $M_1\equiv\sqrt{\frac{2}{3}}(\cos 0,\cos\pi/3,\cos 2\pi/3)$ and $M_2\equiv\sqrt{\frac{2}{3}}(\sin 0,\sin\pi/3,\sin 2\pi/3)$ form a basis of a real two-dimensional plane, which we shall call $S^{(2)}$. Note further that any vector in $S^{(2)}$ is of the form

$$A\cdot M_1+B\cdot M_2, \tag{22}$$

where A and B are constants, and that any normalized vector in $S^{(2)}$ is given by

$$\cos\psi M_1+\sin\psi M_2=\sqrt{\tfrac{2}{3}}(\cos(0-\psi),\ \cos(\pi/3-\psi),\ \cos(2\pi/3-\psi)). \tag{23}$$

The norm $\|I^{(j)||}\|$ of the projection of $I^{(j)}$ into the plane $S^{(2)}$ is given by the maximal possible value of the scalar product $I^{(j)}$ with any normalized vector belonging to $S^{(2)}$, that is

$$\|I^{(j)||}\|=\max_{\psi}\sum_{l_j=1,2,3}I^{(j)}\left(\alpha_j^{l_j},\lambda\right)\sqrt{\frac{2}{3}}\cos\left(\alpha_j^{l_j}-\psi\right)=\sqrt{\frac{2}{3}}\max_{\psi}\mathrm{Re}(z\exp(\mathrm{i}(-\psi))), \tag{24}$$

where $z=\sum_{l_j=1}^{3}I^{(j)}\left(\alpha_j^{l_j},\lambda\right)\exp\left(\mathrm{i}\alpha_j^{l_j}\right)$. We may assume $\left|I^{(j)}\left(\alpha_j^{l_j},\lambda\right)\right|=1$. Then, since $\mathrm{e}^{\mathrm{i}\alpha_j^{l_j}}=\mathrm{e}^{\mathrm{i}[(l_j-1)/3]\pi}$, the possible values for z are $0,\pm 2\mathrm{e}^{\mathrm{i}(\pi/3)},\pm 2\mathrm{e}^{\mathrm{i}(2\pi/3)},\pm 2$. Note that the minimum possible overall complex phase (modulo 2π) of $(z\exp(\mathrm{i}(-\psi)))$ is 0. Then we obtain $\|I^{(j)||}\|\leqslant\sqrt{\frac{2}{3}}\times 2\cos 0=2\sqrt{\frac{2}{3}}$. That is, one has $\|I^{(j)||}\|\leqslant 2\sqrt{\frac{2}{3}}$.

Since M_1 and M_2 are two orthogonal basis vectors in $S^{(2)}$, one has

$$\sum_{l_j=1,2,3}I^{(j)}\left(\alpha_j^{l_j},\lambda\right)\cdot\sqrt{\frac{2}{3}}\cos\alpha_j^{l_j}=\cos\beta_j\|I^{(j)||}\|, \tag{25}$$

and

$$\sum_{l_j=1,2,3}I^{(j)}\left(\alpha_j^{l_j},\lambda\right)\cdot\sqrt{\frac{2}{3}}\sin\alpha_j^{l_j}=\sin\beta_j\|I^{(j)||}\|, \tag{26}$$

where β_j is some angle. Using this fact one can put the value of (12) into the following form:

$$\left(\sqrt{\frac{3}{2}}\right)^N\prod_{j=1}^{N}\|I^{(j)||}\|\times\sum_{i_1,i_2,\ldots,i_N=1,2}T_{i_1i_2\ldots i_N}d_1^{i_1}d_2^{i_2}\ldots d_N^{i_N}, \tag{27}$$

where

$$\left(d_j^1,d_j^2\right)=(\cos\beta_j,\sin\beta_j). \tag{28}$$

Let us look at the expression

$$\sum_{i_1 i_2 \ldots i_N = 1,2} T_{i_1 i_2 \ldots i_N} d_1^{i_1} d_2^{i_2} \ldots d_N^{i_N}. \tag{29}$$

Formula (28) shows that we deal here with two-dimensional unit vectors $\vec{d}_j = \left(d_j^1, d_j^2\right)$, $j = 1, 2, \ldots, N$, that is (29) is equal to $\hat{T} \cdot (\vec{d}_1 \otimes \vec{d}_2 \otimes \cdots \otimes \vec{d}_N)$, i.e. it is a component of the tensor $\hat{T}$ in the directions specified by the vectors $\vec{d}_j$. If one knows all the values of $T_{i_1 i_2 \ldots i_N}$, one can always find the maximal possible value of such a component, and it is equal to $T_{\max}$, of equation (10).

Therefore since $\|I^{(j)||}\| \leqslant 2\sqrt{\frac{2}{3}}$ the maximal value of (27) is $2^N T_{\max}$, and finally one has

$$(E_{\mathrm{LR}}, E) \leqslant 2^N T_{\max}. \tag{30}$$

Please note that relation (30) is a generalized Bell inequality. Specific local realistic models, which predict three-setting models, must satisfy it. In the following section, we shall show that if one replaces E_{LR} by E one may have a violation of the inequality (30). One has

$$\begin{aligned}(E, E) &= \sum_{l_1=1,2,3} \sum_{l_2=1,2,3} \cdots \sum_{l_N=1,2,3} \left(\sum_{i_1, i_2, \ldots, i_N = 1,2} T_{i_1 i_2 \ldots i_N} c_1^{i_1} c_2^{i_2} \cdots c_N^{i_N} \right)^2 \\ &= \left(\frac{3}{2}\right)^N \sum_{i_1, i_2, \ldots, i_N = 1,2} T_{i_1 i_2 \ldots i_N}^2. \end{aligned} \tag{31}$$

Here, we have used the fact that $\sum_{l_j=1,2,3} c_j^{i_1} c_j^{i_1'} = \frac{3}{2} \delta_{i_1 i_1'}$, because $c_j^1 = \cos \alpha_j^{l_j}$ and $c_j^2 = \sin \alpha_j^{l_j}$.

The structure of condition (30) and the value (31) suggests that the value of (31) does not have to be smaller than (30). That is there may be such correlation functions E, which have the property that for any E_{LR} (three-setting model) one has $(E_{\mathrm{LR}}, E) < (E, E)$, which implies impossibility of modeling E by the 'three-setting' local realistic correlation function E_{LR} with respect to the three measurement directions.

3. The difference between the two-setting model and three-setting model

We present here an important example of a violation of (30). It presents the difference between the two-setting model and three-setting model. Imagine N observers who can choose between two orthogonal directions of spin measurement, $\vec{x}_j^{(1)}$ and $\vec{x}_j^{(2)}$ for the jth one. Let us assume that the source of N entangled spin-carrying particles emits them in a state, which can be described as a generalized Werner state, namely $V|\psi_{\mathrm{GHZ}}\rangle\langle\psi_{\mathrm{GHZ}}| + (1 - V)\rho_{\mathrm{noise}}$, where $|\psi_{\mathrm{GHZ}}\rangle = 1/\sqrt{2}(|+\rangle_1 \cdots |+\rangle_N + |-\rangle_1 \cdots |-\rangle_N)$ is the Greenberger, Horne and Zeilinger (GHZ) state [14] and $\rho_{\mathrm{noise}} = \frac{1}{2^N} \mathbb{1}$ is the random noise admixture. The value of V can be interpreted as the reduction factor of the interferometric contrast observed in the multi-particle correlation experiment. The states $|\pm\rangle_j$ are the eigenstates of the σ_z^j observable. One can easily show that if the observers limit their settings to $\vec{x}_j^{(1)} = \hat{x}_j$ and $\vec{x}_j^{(2)} = \hat{y}_j$ there are 2^{N-1} components of $\hat{T}$ of the value $\pm V$. These are $T_{11\ldots1}$ and all components that except from indices 1 have an even number of indices 2. Other x–y components vanish.

It is easy to see that $T_{\max} = V$ and $\sum_{i_1, i_2, \ldots, i_N = 1,2} T_{i_1 i_2 \ldots i_N}^2 = V^2 2^{N-1}$. Then, we have $(E_{\mathrm{LR}}, E) \leqslant 2^N V$ and $(E, E) = (\frac{3}{2})^N V^2 2^{N-1} = \frac{3^N}{2} V^2$. For $N \geqslant 6$, and V given by

$$2\left(\frac{2}{3}\right)^N < V \leqslant \frac{1}{\sqrt{2^{N-1}}} \tag{32}$$

despite the fact that there exist 'two-setting' local realistic models for three measurement directions in consideration $\left((0, \frac{\pi}{3}, \frac{2\pi}{3}) \equiv (A, B, C)\right)$, these models cannot construct 'three-setting' local realistic models. Namely, even though there exist two-setting models for a set of measurement directions (A, B), (B, C) and (C, A), these models cannot construct any three-setting models for (A, B, C).

As it was shown in [13] if the correlation tensor satisfies the following condition:

$$\sum_{i_1, i_2, \ldots, i_N = 1,2} T^2_{i_1 i_2 \ldots i_N} \leqslant 1 \tag{33}$$

then there always exists a 'two-setting' local realistic model for the set of correlation function values for all directions lie in a plane. For our example condition (33) is met whenever $V \leqslant \frac{1}{\sqrt{2^{N-1}}}$. Nevertheless, such models cannot construct 'three-setting' local realistic models for $V > 2\left(\frac{2}{3}\right)^N$. Thus the situation is such for $V \leqslant \frac{1}{\sqrt{2^{N-1}}}$ for all two-settings-per-observer experiment one can construct the 'two-setting' local realistic model for the values of the correlation function for the settings chosen in the experiment. One wants to construct the 'three-setting' local realistic model for three measurement directions (A, B, C) using 'two-setting' local realistic models, (A, B), (B, C) and (C, A). But these three 'two-setting' models must be consistent with each other, if we want to construct truly 'three-setting' local realistic models beyond the 2^N settings to which each of them pertains. Our result clearly indicates that this is impossible for $V > 2\left(\frac{2}{3}\right)^N$. These 'two-setting' local realistic models, (A, B), (B, C) and (C, A) must contradict each other. Rather they are therefore invalidated. In other words the explicit two-setting models, given in [13] work only for the specific set of settings in the given experiment, but cannot construct a local realistic model for the values of a correlation function, given in a three-setting Bell experiment (three-setting model), even though there exist two-setting models for the three measurement directions chosen in the given three-setting experiment.

One can see that the three-setting model (even if exists) does not have the property which the plane-infinite-model has when $2\left(\frac{2}{3}\right)^N > V > 2\left(\frac{2}{\pi}\right)^N$ [11]. Thus, the three-setting model is different from the plane-infinite-model.

Please note that all information needed to get this conclusion can be obtained in a two-orthogonal-setting-per-observer experiment, that is with the information needed in the case of 'standard' two-settings Bell inequalities [13, 15–17]. To get both the values of (31) and of $T_{\max}$ it is enough to measure all values of $T_{i_1 i_2 \ldots i_N}$, $i_1, i_2, \ldots, i_N = 1, 2$.

4. Summary

In summary we derived a generalized Bell inequality for N qubits which involves three-setting for each of the local measuring apparatuses. The inequality forms a necessary condition for the existence of a local realistic model for the values of a correlation function, given in a three-setting Bell experiment. And we have shown that a local realistic model for the values of a correlation function, given in a two-setting Bell experiment, cannot construct a local realistic model for the values of a correlation function, given in a three-setting Bell experiment, even though there exist two-setting models for the three measurement directions chosen in the given three-setting experiment. Hence the property of the two-setting model is different from the property of the three-setting model.

Our result provided classification of local realistic theories. At least, we can see four types of models. First, there is the two-setting model. It is explicitly constructed. However, this model is disproved by several generalized Bell inequalities. The patterns of the disqualification are different to each other. Therefore, one furthermore has three different types of models. These are the three-setting model, plane-infinite-model and sphere-infinite-model.

How does our Bell inequality help us to understand certain quantum protocols? What can it be used for? We leave these questions as an open question.

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